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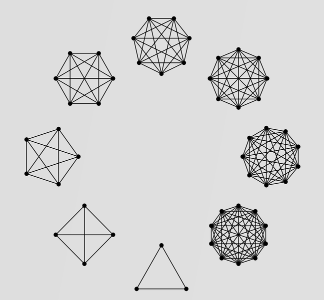


Minimizing Direct Competitions in Complete Components of (1,2)-Step Competition Graphs

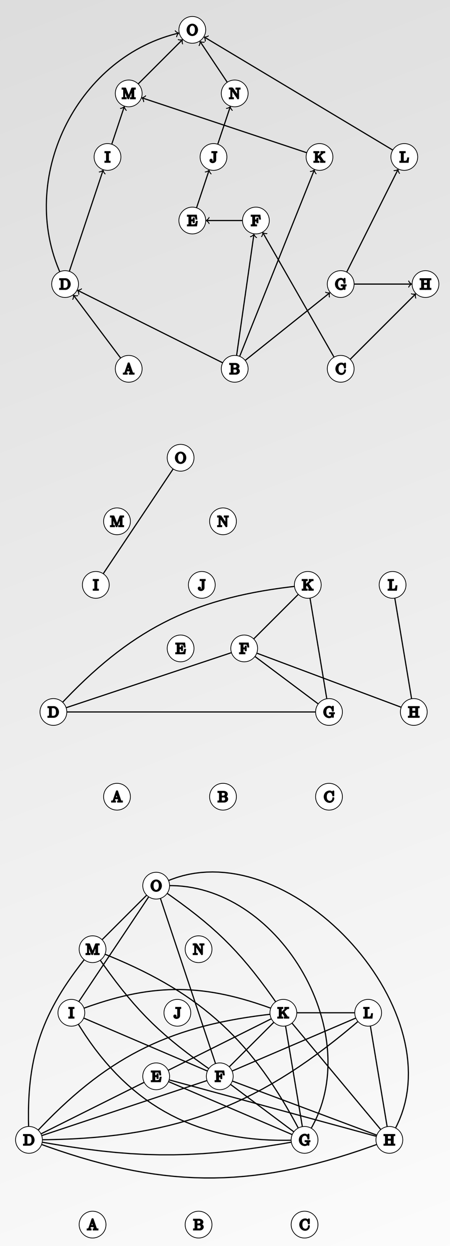


Figure 1. an example food web and its competition graph and (1,2)-step competition graph.

Introduction

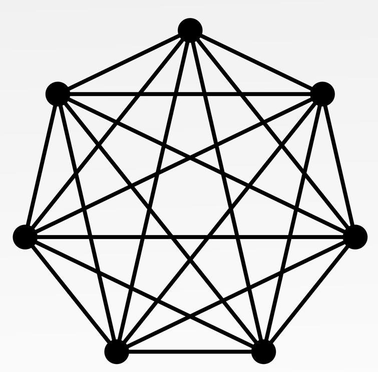
•Graph theory is a useful tool for studying systems of food webs, a concept from ecology that models the predator-prey relationships between species in an ecosystem.

•We use the concept of food webs to inform and motivate our exploration of graph theory.

•In particular we examine the characteristics of (1,2)-step competition graphs, which are an extension of normal competition graphs (Factor and Merz, 2010).

•In general, when considering digraphs that represent real food webs, we place some restrictions on the types of graphs that we examine. Specifically we limit our investigations to simple, acyclic digraphs that are weakly connected (Pimm et al., 1991).

•Of particular interest are complete graphs, graphs that have an edge between each pair of vertices. During the course of our research we developed a family of digraphs that induce a complete graph as a component of their (1,2)-step competition graphs. This group of digraphs appear to induce a minimum number of direct competitions (see definitions below) in these complete components.



A complete graph on 7 vertices.

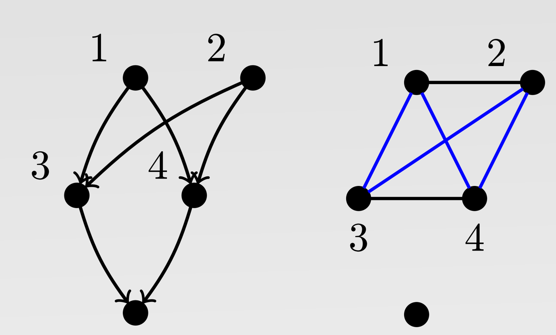
Background & Definitions

* **Definition.** *Given a digraph D, the competition graph of D, denoted $C(D)$, is the graph with the same vertex set as D and an edge between vertices x and y if and only if $O^+(x) \cap O^+(y) \neq \emptyset$.* (Factor and Merz, 2010)

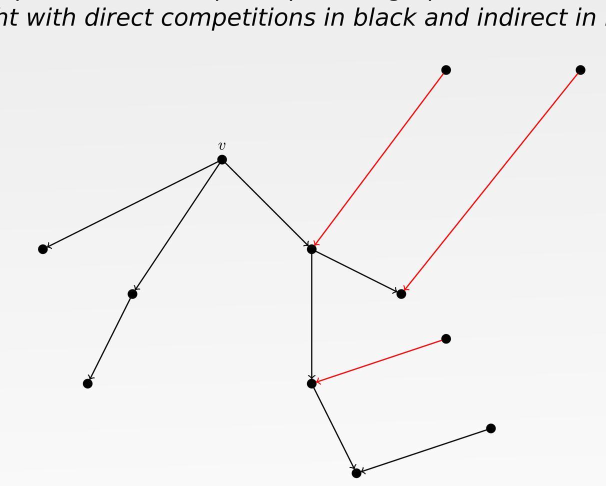
* **Definition.** *Given a digraph D, the (1,2)-step competition graph of D, denoted $C_{1,2}(D)$, is the graph with the same vertex set as D and an edge between vertices x and y if and only if there exists some $z \in V(G)$ for which either $d_{D\setminus\{x\}}(y,z) = 1$ and $d_{D\setminus\{y\}}(x,z) \leq 2$ or, $d_{D\setminus\{y\}}(x,z) = 1$ and $d_{D\setminus\{x\}}(y,z) \leq 2$.* (Factor and Merz, 2010)

* **Definition.** *Let $G = (V,E)$ be the (1,2)-step competition graph of a digraph $D = (V,A)$. If $x, y \in V(D)$ (1,2)-compete for a vertex z in D such that $d_{D\setminus\{x\}}(y,z) = 1$ and $d_{D\setminus\{y\}}(x,z) = 1$, then we say that x and y* ***compete directly*** *and we call the edge $(x,y) \in E(G)$ a direct competition between x and y. If x and y (1,2)-compete for a vertex z in D such that $d_{D\setminus\{x\}}(y,z) = 1$ and $d_{D\setminus\{y\}}(x,z) = 2$, then we say that x and y* ***compete indirectly*** *and we call the edge $(x,y) \in E(G)$ an indirect competition between x and y.*

* **Theorem.** *Let $G = (V,E)$ be the (1,2)-step competition graph of a digraph $D = (V,A)$. A vertex $v \in V$ is isolated in G if and only if for every $u \in O^+(v)$ the following conditions hold:*
  1. $O^-(u) = \{v\}$ *and*
  2. *for all* $y \in O^+u$, $O^-(y)\setminus\{u\} = \emptyset$.



An example of a digraph that induces a complete component on 4 vertices in its (1,2)-step competition graph. The (1,2)-step competition graph is shown to the right with direct competitions in black and indirect in blue.



An example of a vertex v that would be isolated in the corresponding (1,2)-step competition graph. If the arcs colored in red are present in the digraph, then they cause the vertex v to compete with other vertices in the graph

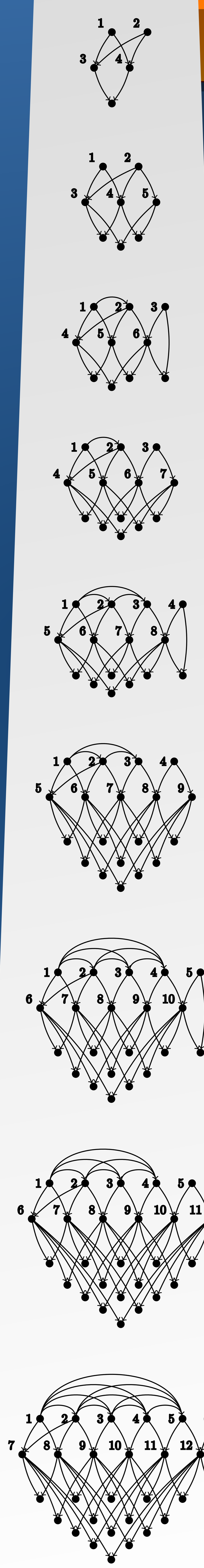


Figure 2. digraphs that induce complete components in their (1,2)-step competition graphs.

Results

•We constructed the family of digraphs shown in fig. 1 in an effort to the find the minimum number of direct competitions that can be induced in a complete component of a (1,2)-step competition graph.

* **Theorem.** *The family of digraphs shown in fig. 2 each induce K_n on the n numbered vertices as a component of their (1,2)-step competition graphs.*

* **Lemma.** *For $n \geq 10$, the complete components that the digraphs from fig. 2 induce in their (1,2)-step competition graphs have exactly $\frac{n^2-6n+24}{4}$ direct competitions for n even and $\frac{n^2-6n+25}{4}$ direct competitions for n odd.*

* **Theorem.** *For any digraph D that is a member of the family of digraphs shown in fig 2. where $n \geq 10$, the ratio of direct competitions to total edges in the component K_n of $C_{1,2}(D)$ is given by:*

$$\frac{n^2-6n+24}{2n^2-2n} \tag{1}$$

for n even and

$$\frac{n^2-6n+25}{2n^2-2n} \tag{2}$$

for n odd. Furthermore, eq. (1) and eq. (2) are both increasing sequences for $n \geq 10$, and

$$\lim_{n\to\infty} \frac{n^2-6n+24}{2n^2-2n} = \lim_{n\to\infty} \frac{n^2-6n+25}{2n^2-2n} = \frac{1}{2}.$$

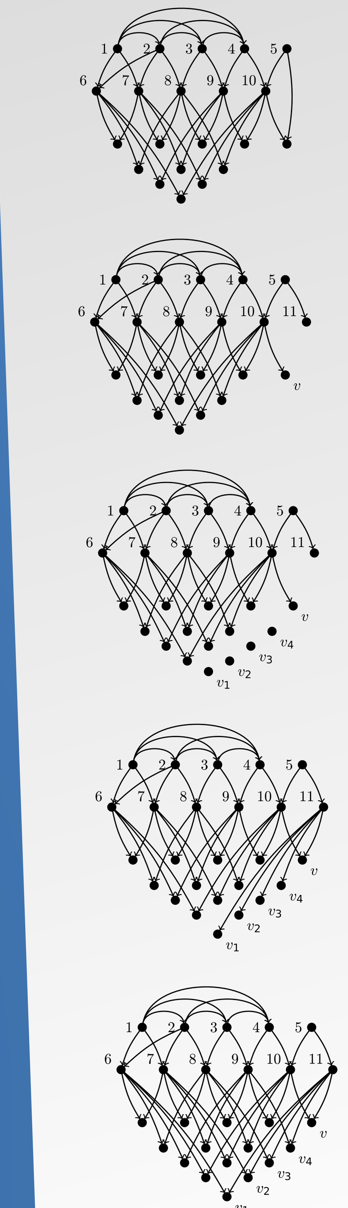


Figure 3. an example of the constructive process used to generate the digraphs in fig. 2.

Future Work

•We conjecture that the family of digraphs shown in fig. 2 induce the minimum possible number of direct competitions in a complete component of a (1,2)-step competition graph. We have yet to find a proof or a counter example to this.

•There are some results that still need to be written up and proven rigorously. There exist some superfluous arcs in the digraphs in fig. 2 that are not necessary in order to induce a complete component.

References & Acknowledgments

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